Senior Project

Delimit Service Scope of Power Plants by Weighted Voronoi Diagram

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Abstract The paper will present an application of Voronoi diagram on improving power distribution efficiency by delimiting geographically service scope of power plants in the U.S. A weighted 2D Voronoi diagram will be constructed based on factors such as the capacity of the power plants (the maximum electric output a power plant can produce), the location of the power plants, etc. Delimiting an appropriate service scope of each power plant is crucial for energy efficiency, because the longer the distance of transmission, the higher the energy loss would be. In the end, Voronoi diagrams will be constructed and visualized with the idealized service scope of each power plant.

Acknowledgments

Here, you can thank whoever you wish to thank.

Contents

1   Introduction

Electricity distribution and transmission efficiency has been a crucial consideration when constructing the electricity delivery system. A main factor that contributes to the inefficiency is energy loss during long distance transmission [Mar19]. In the United States, modern electricity transmission systems have driven toward larger scale and more centralized electricity generation utilities, in contrast to local, small electric utilities that were dominant before the early 20th century [Age]. That is, the electricity that flows to the consumer is usually generated from a distant power station and travels along the long transmission lines across cities, even states [Age]. Although the centralized layout has increased supply reliability, the drawback is also salient – the transmission cost could be up to 30% more than the distributed layout [Mar19]. "Line loss", the loss caused by the resistance when the electricity flows through the cable, contributes to the main energy loss, and the loss increases proportionally with the transmission distance, which can be up to $10\% - 20\%$ of the total electricity transmitted [Ele].

Therefore, an ideal electricity distribution model should allocate each demand point to the nearest power generator possible without exceeding each generator's maximum energy output, the capacity of the generator. To construct such a distribution model, both the traditional Voronoi Diagram(VD) and weighted VD are applied to a dataset of over 8000 power plants in the United States, including each of their location and capacity. By constructing VD, the territory is partitioned into regions, and each region will be the optimal service scope of a power plant; that is, the power plant will be the main the power source for all demand points within the region. The traditional VD can minimize transmission loss by assigning each demand point to the closest possible power point; however, it does not consider practical factors such as the capacity of power plants, which can widely vary from one power plant to another. The weighted VD incorporates the capacity as weight to reduce potential short supply and avoid oversupply while still maintain VD's property of capturing proximity. We will further introduce VD's properties and applications in the following

section.

2 Background

2.1 Introduction to Voronoi Diagram

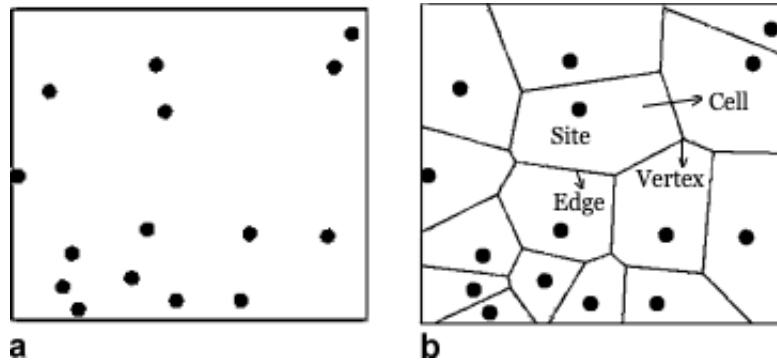### 2.1.1 Voronoi Diagram and Properties

Figure 1: An example of VD creation [Kan08].

The traditional VD is a partition of plane based on a set of points. Those points are referred to as "sites" or "generators". Each site has one corresponding region called "cell". All the points in a cell are closer to the corresponding site than to any other site. The points that are equidistant from two sites form a Voronoi edge, and the points that are equidistant from more than two sites are Voronoi vertices. The following is some of the properties of VD:

1. All Voronoi regions are convex.

2. Voronoi cells are mutually exclusive except for boundaries.

3. The union of all Voronoi cells covers the entire plane.

4. The generator g is the nearest generator point from point p if and only if the corresponding cell of g contains p.

5. For any 2 adjacent Voronoi sites, the sites have the same distance to the edge between them.

6. The straight-line dual graph of VD corresponds to Delaunay Triangulation, which will be introduced in the later section.
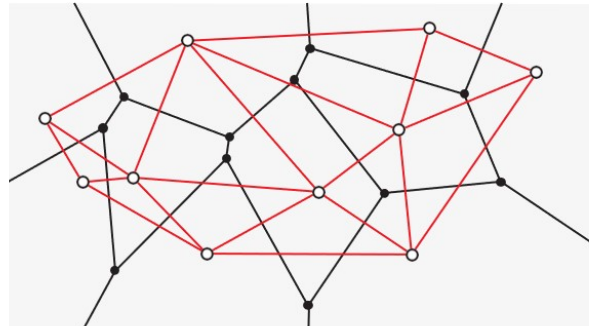

Figure 2: Dual of VD: connecting sites if they share a common Voronoi edge. The result is a Delaunay triangulation (in red) [DO11].

2.1.2 Delaunay Triangulation and Properties

Given a set of points S, a convex hull is outlined in blue as shown in graph below. A triangulation of S is a partition of the convex hull. Each subdivision (face) is a triangle, whose vertices are the points and does not contain other points.
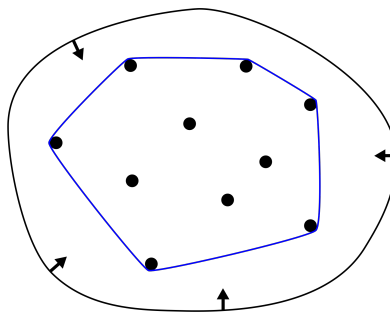

Figure 3: The convex hull of a point set: the region outlined by blue segments [Wik21]

There are exponential number of triangulations given a point set. Delaunay triangulation (DT) is one special kind of triangulation where the circumcircle of each triangle does not contain other points, known as "the empty circle property".
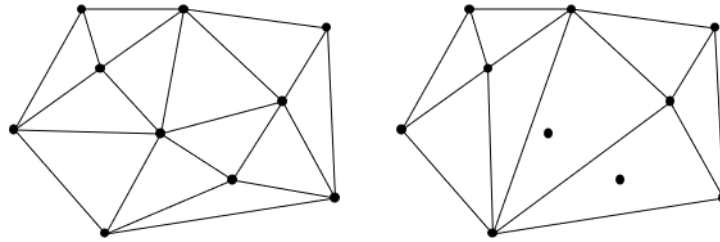
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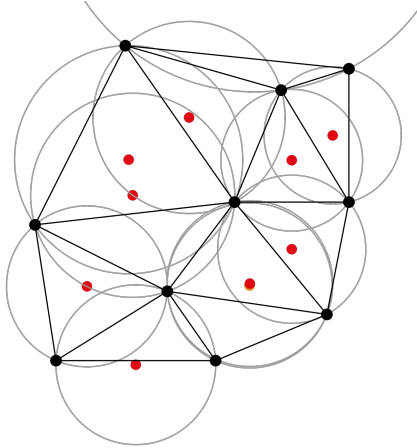Figure 4: (a) Point set triangulation. (b) Not a triangulation [ETH13].

Figure 5: A DT with circumcircles(in grey) and circumcenters(red dots). Note that all circles are empty of points(black dots) [con21].
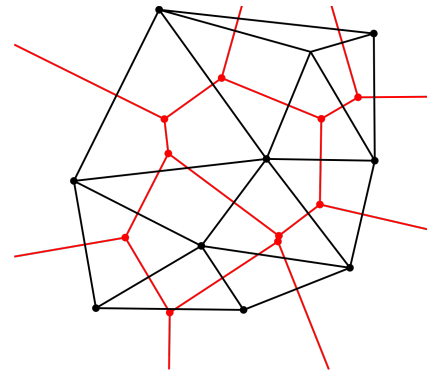
Figure 6: Connecting the centers of circumcircles produces a VD [con21].

As mentioned above, an important feature of DT is its duality with VD. Figure 2 and 6 show the transformation from one to the other. "The one-to-one correspondence provided by duality" suggests that VD can be computed via DT [DO11]. In section 4, we will further discuss different algorithms of VD construction.

Another important property of DT is that it tends to avoid "skinny" triangles, which are triangles with one or two extremely acute angles and hence have a long, thin shape. This property makes DT more desirable in some circumstances. For example, in terrain construction, DT can better capture the feature of land surfaces such as ridges and valleys compared to other triangulations [DO11].

The generalized Delaunay triangulation is Regular triangulation(RT), which is defined for a set of weighted points [BDP+02]. RT is the dual of the power diagram, which will be
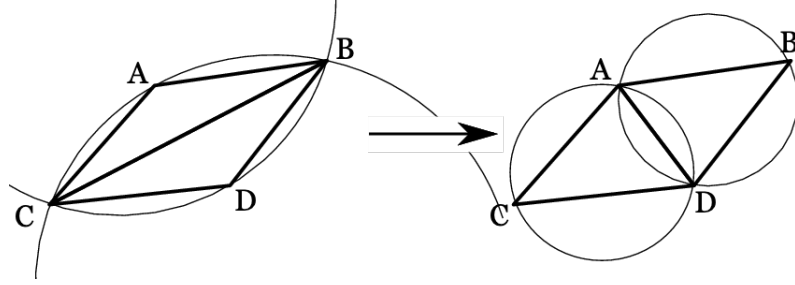
Figure 7: (a) Non DT has long, thin triangles (b) DT has "fat" triangles [Flo16].

introduced in the next section.

2.1.3 Power Diagram

Power diagram(PD) is one of the weighted VD where sites are assigned with weights to reflect variation in attribute of sites. The sites are inflated into circles with radius of $\sqrt{w}$, where w is the weight of a site. PD is a partition of plane where each point falls into the region of the nearest site, however, with power distance instead of Euclidean instance as measurement. The power distance between a given point P and some site O is now a tangent segment to a circle that centered at O, which is PT in the diagram below.

$$PA \cdot PB = PT^2 = d^2 - r^2$$

Figure 8: The distance metrics of Power Diagram [Wik20]

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2.2 Algorithms for Computing Voronoi Diagrams

One way of constructing the VD is via DT. One computes DT first and transforms the solution to its dual VD. A straight forward DT solution is the flip algorithm. It starts with a random triangulation and keeps "flipping" edge, as shown in figure 7, u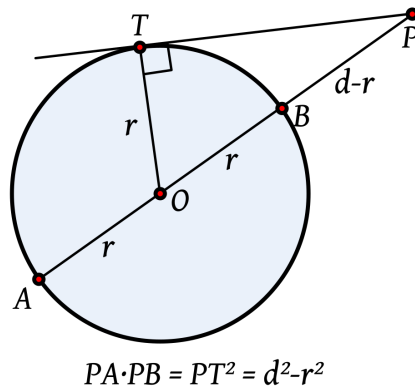ntil all the triangles satisfy empty circle property. Other DT algorithms includes Bowyer-Watson algorithm, DeWall algorithm, and Quickhull algorithm.

Similarly, a power diagram can be generated via RT.
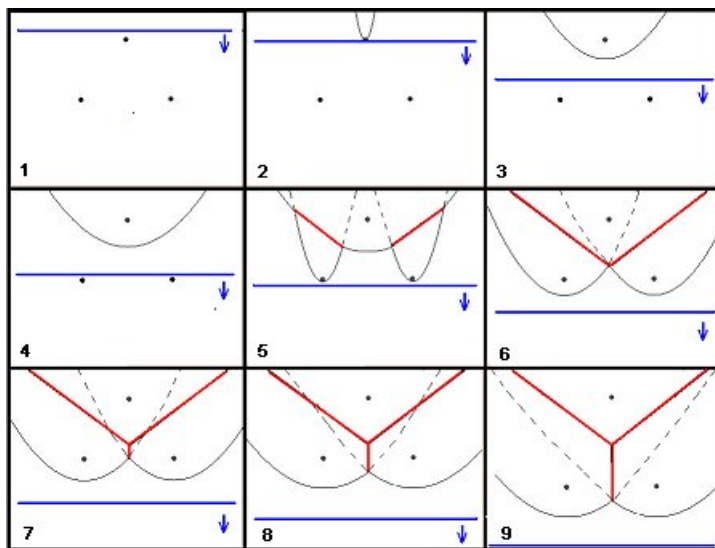
Figure 9: A demo of sweep line algorithm showing how VD (in red) is traced out by break points [Tah06].

For direct construction of VD, there are various approaches, including the incremental approach, the divide-and-conquer approach and Fortune's algorithm (the sweep line algorithm). The incremental approach proposed by Peter Green and Robin Sibson in 1977 constructs the diagram by including one site a time to the current diagram and form a new cell, until all sites are included. The time-complexity is $O(n^2)$ [DO11]. The divide-and-conquer approach recursively divides the plane into two halves and construct the diagram for each half, and then combines the two diagrams together. The time complexity is $O(nlogn)$, which is more efficient but the combination step "is complex and hard to implement" [For86]. Fortune's sweep line has the same time complexity as the divide-and-conquer approach but a less com-

plicated implementation. The algorithm maintains a horizontal line sweeping from top of the plane to the bottom, which is defined as "sweep line". The sweep line and the sites above co-determine a series of parabolas. The union of the parabolas forms the "beach line", which changes as the sweep line moves. The changes on beach line give information on Voronoi egdes and vertices. The breakpoints on the beach line trace out the Voronoi edges. The disappearance of an parabola generates a Voronoi vertex. When the sweeping ends, the VD is produced [For86].
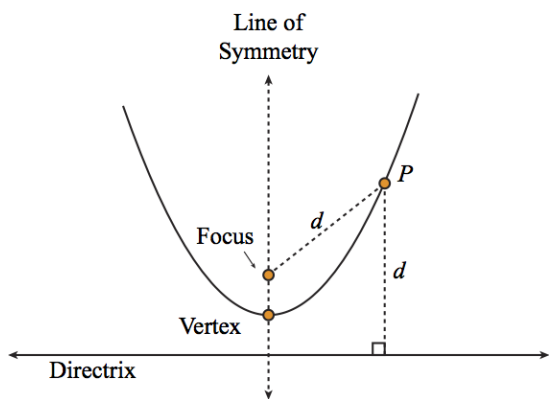
Figure 10: A point (focus) and a line (directrix) defines a parabola [shm].
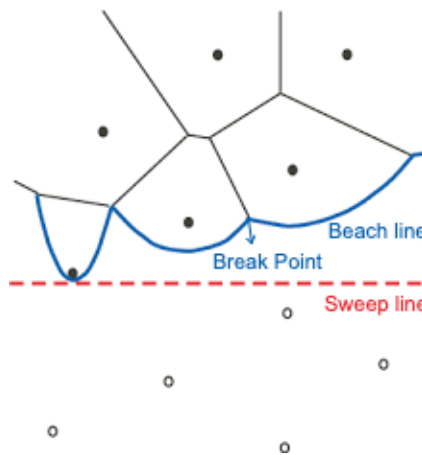
Figure 11: Static visualization of Fortune's algorithm.

2.3   Applications of Voronoi Diagram

The simplicity and elegance of concept and its nature of capturing proximity has made VD widely used and applied to solve various types of problems [DO11]. In particular, VD has major applications in solving proximity, similarity and coverage problems. For proximity, VD helps to find the nearest facilities such as hospitals or post office [Tra21], and it also contributes to solving shortest-path finding problems [ADMS13] and k-closest-neighbor problem [Kol04]. For similarity, VD has been applied to objects matching [DMDG15], categorization [SPR19] and clustering [KRE06]. For coverage, VD has been used for facility layout planning [LSL16] and optimize the service scope of facilities [ZJC18].

In addition, VD has extensive application in a variety of disciplines including archaeology, meteorology, biology, epidemiology, etc. In archaeology, VDs are often used to define the domain of influence of neolithic clans or hillforts [OBS09]. In biology, Voronoi tessellation helps to model the 3D bone microstructure [LLK$^+$12], contributing to the measurement of physical constraints of body tissues. In meteorology, VDs, referred to as Thiessen polygon [Sch98], are used to compute average areal rainfall of a catchment from discrete data collected by rain gauges. In epidemiology, a famous application is the 1854 London Broad Street cholera outbreak [oE]. A VD is constructed to accurately delineate the region where the most deaths clustered and successfully identified the source of infection [oE].

Moreover, various generalizations and extensions have been developed to resolve more complex problems or better adapt to real-world situations [OBS09]. For example, the high-order VD(or k-order VD) has more than 1 points that constitute of a generator [OBS09]. The k-order VD has been applied to facility location problems in terms of determining the critical k nearest facilities [OBS09]. Another important generalization is the weighted VD. The traditional VD implicitly assumes that sites are equally weighted. The weighted VD takes the variation in the properties and attributes of sites into account, which could be more suitable for practical applications [FM18] such as economic markets modeling, delivery system design, functional territories for facilities, etc. [OBS09].

3   Related Work

There is a large body of work using VD to compute spatial coverage [OBS09]. We will focus on discussing studies that use weight VD to solve coverage problems because which is the approach we will be using ourselves.

Weighted VD has been commonly to resource allocations and facility layout design. For example, a study on determining ecosystem service scope [ZJC18] compares the accuracy of traditional VD with the weighted VD. Since ecosystem service value is a crucial indicator of

the strength of ecosystem and varies from region to region, the weighted diagram incorporates ecosystem service value and results in a more scientific service area delineation than the traditional approach does. The weighted VD has also been applied in a study of modeling water distribution networks [GK12]. The objective is to assign water demand points to the closest water sources. Being aware of the geographical unevenness, a weighted VD is constructed with an additive constant on obstacles and boundaries that could not be passed by distribution pipes. In addition, weighted VD has also been used to optimize the layout of urban system [XSM+10] and rural settlements [WLX+20], to choose a new site for chain stores [LSL16], etc.

Moreover, the weighted VD has also been largely applied to emergency modeling and rescue route optimization, which usually adapts to the real-world circumstances better than the traditional VD [FM18]. For example, a study in delimiting the service area of fire emergency stations has the fire risk index as the weight incorporated into distance calculation. In addition, instead of the Euclidean distance, Manhattan distance is adopted to mimic the actual rescue routes [YCC+20]. Another study on drone delivery of EMS from hospitals compares the traditional VD with the weighted VD that incorporates factors such as wind magnitude and obstacles. As a result, critical amount of response time, which is the most important measurement for delivery efficiency, could be saved by using the weighted VD [FM18]. Similarly, in a study on locating distribution center of disaster supplies, a weighted VD is applied to maximize the total coverage area of the distribution centers to produce a quick response time and minimize human suffering [YJU12] Moreover, weighted VD also helps with building an online map-based emergency management support system [LLTB11].

Thus, from the few studies discussed above, we see that weighted VD has been widely applied to solving coverage problems, which can compensate for spatial unevenness and produce more scientific results.

In this paper, we are going to apply a weighted VD on delimiting the service scope of power plants in the United States in order to minimize the energy loss in transmission. The

weight is introduced to reflect the variation in power plant capacity. Though we have not found any direct VD application on service scope delimitation for power plants, there is a relevant study on applying weighted VD to determining new substation sites, substation capacity and service scope to minimize the overall annual cost [FLZY]. The weight has been used to reflect the variance in load distribution and capacity. The following is the major differences between our study and theirs:

1. In our study, the object is power plants which generate electricity, whereas their object is substations which distribute electricity.

2. Our study determines the service scope of located power plants, whereas they determines the location of new substations and their capacities.

3. Our objective is to reduce energy loss in transmission, and theirs is to reduce the overall substation operation cost.

To our knowledge, this is the first attempt to reduce electricity loss in transmission by delimiting service scope for power plants. We have not found the information of the existing service scope of power plants within the U.S., so we are not able to compare our design with the existing service scope. We are also aware that the existing service scope may have taken other factors such as terrain, population density into consideration, whereas our model is idealized and simplified that is only based on power plant location and capacity.

4   Methodology

In order to determine each power plant's optimal service scope, we construct and compare the following two diagrams:

1. A traditional VD, where each Voronoi site represents a power plant. The distance measure used is the normal Euclidean distance.

2. A weighted VD(power diagram), where each Voronio site has a radius of the square root of capacity. The distance measure used is power distance.

4.1 Data Description and Preprocessing

The data used for the application is obtained from a comprehensive, open source database of the power plants all over the world. The database is developed by World Resources Institute (WRI) in partnership with Google Earth Outreach and multiple authoritative organizations and it mainly draws upon trusted sources such as national governments and other official sources, integrated with a small portion of crowdsourced data such as satellite images for accuracy improvement [BFH+19]. More importantly, among all current existing databases on this topic, it's the only one "truly comprehensive and fully accessible" [BFH+19] with all sources traceable to publicly available webpages, and is still under maintenance. The dataset in CSV form can be accessed at https://www.kaggle.com/eshaan90/global-power-plant-database. We adapt the latest official release, the release of 2018.

The dataset contains over 28,500 power plants in 164 countries, representing about 80% of the world's capacity [BFH+19]. The set of attributes of power plants covers identification information (name and unique identifier), geographic information (country, longitude and latitude in WGS84 standard), electricity generation data (generation capacity in megawatts, actual generation for each year from 2013 to 2016, estimated annual electricity generation), and other information including owner of the plant, year of operation, attribution of data sources.

We leverage Python pandas library for exploratory data analysis and wrangling. Since our study object is the power plants within the United States, we only keep the dataset entries that are within the U.S. The processed dataset has 8119 instances with 5 attributes: ID, name, longitude, latitude and capacity. Through the data profiling, there are no missing data, and have reasonable range of numbers. The electrical generation capacity ranges from 1 to 6800 megawatts, the longitude ranges from -171.71 to 144.90, the latitude ranges from

13.30-71.29. All figures are within reasonable ranges.

4.2   Tools and Packages

We construct the traditional VD by using D3-Voronoi package and visualize by Google Maps JavaScript API. D3-Voronoi is a package written in JavaScript for computing 2D-VD. We choose D3-Voronoi because it implements Fortune's algorithm and also has decent visualization (e.g. supports zoom transitions) [Riv19]. The shortcoming is that the package itself does not support constructing more complex VD, including weighted VD, but it is satisfactory enough for constructing the traditional VD. Google Maps JavaScript API supports different types of map creation(roadmap, satellite, hybrid, and terrain), which are able to display on web pages and mobile devices [Gooa].

The power diagram is computed by using the Computational Geometry Algorithms Library (CGAL), which is a C++ library of robust geometry algorithms. The power diagram is produced by constructing its dual diagram, a RT. Then, we use Processing, a graphical library and IDE, to visualize the diagram computed by CGAL. We use Google Static Map API to obtain static map image as the canvas for drawing.

5   Results and Analysis

In this section, we are going to compare the service scope delimitation produced by traditional VD and by the weighted VD(power diagram) and seek for a more reasonable and scientific solution.

5.1   Traditional Voronoi Diagram and Visualization

Gaining insight from [Shi], traditional VDs are generatedshown in Figure 9 and 10. As we can see, the territory of United States is partitioned into small cells by the red lines, and within each cell there is one black dot, which represents a power plant. The area covered by

Figure 12: Service Scope Delimitation Visualization by Traditional VD

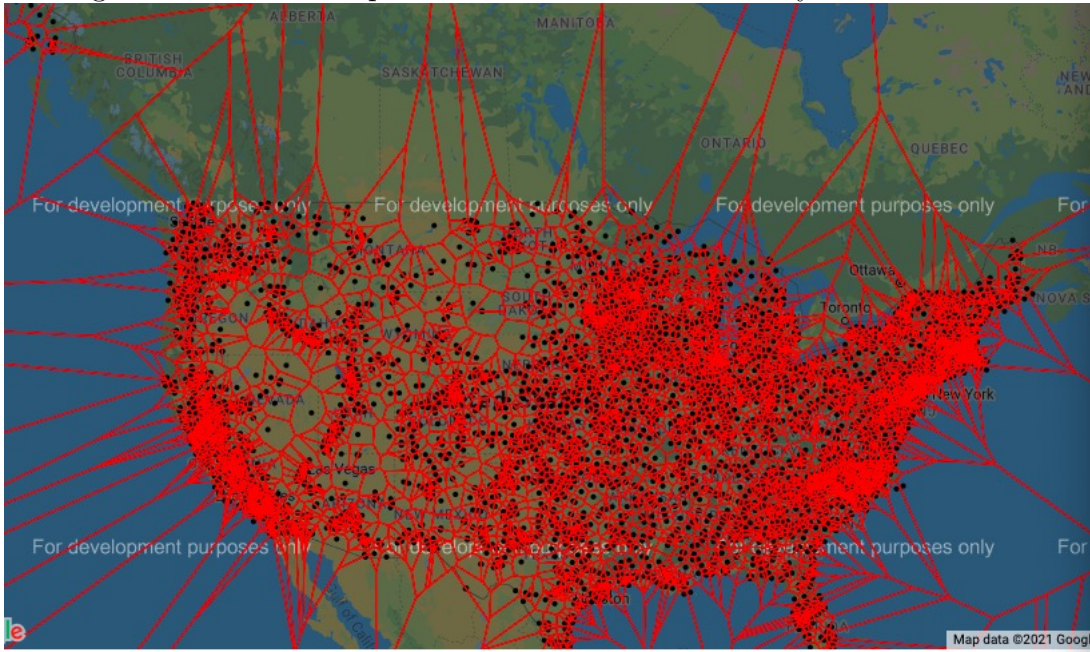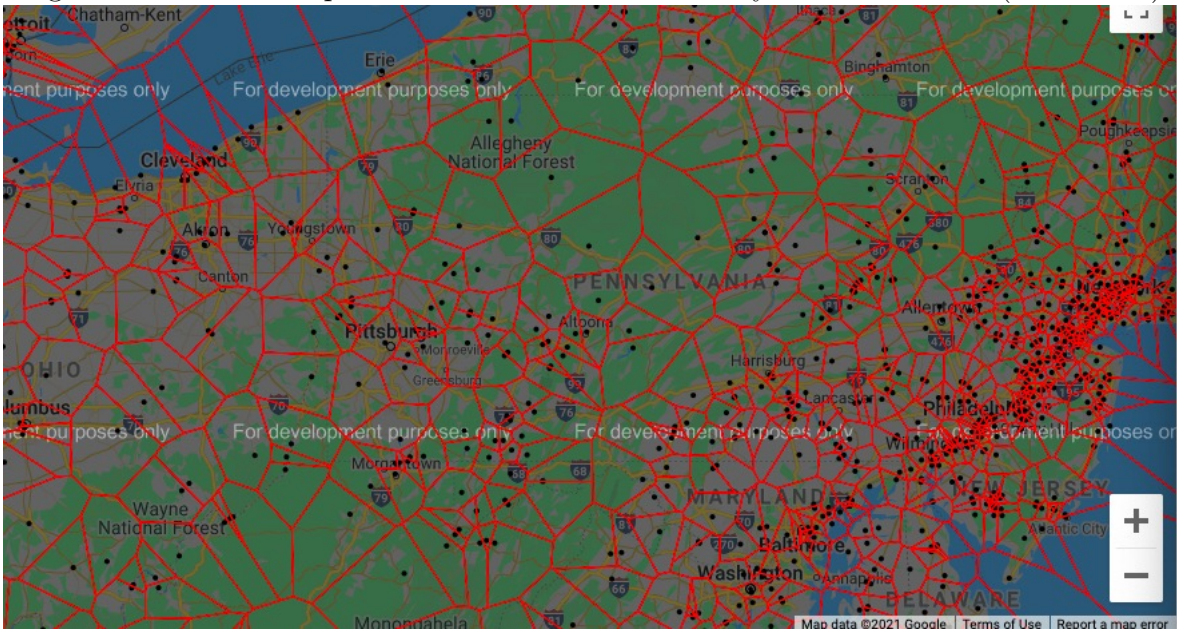Figure 13: Service Scope Delimitation Visualization by Traditional VD (zoomed-in)

each cell is optimal service scope for the power plant in the cell.

This approach simply "assigns" every point in the territory to the closest power plant in Euclidean distance, regardless the different attributes the power plants have in reality. By the property of traditional VD(referring to property 5 in section 2.1.1), each edge has the

same distance to the two adjacent sites.

5.2   Power Diagram and Visualization

5.2.1   Weight Tuning

We incorporate the capacities of the power plants as weight to construct the power diagram. The capacities of the power plants ranging from 1 to 6809 megawatts. The mean capacity is about 711 megawatts, and the median is about 12 megawatts. Thus, the data not only has a wide range, but also heavily right-skewed. Therefore, we take the natural logarithm to the capacity values and then apply the min-max normalization, and now the weight ranges from 0 to 1 with mean = 0.32 and median = 0.28.

5.2.2   Visualization and Comparison

We construct PD on the scope the entire U.S. territory. Figure 14 is the zoomed-in visualization that shows the delimitation of an area centered around the Grand Coulee Dam(GCD) power plant, and we will focus on that area for analysis. GCD is the largest power plant in the U.S. with the largest capacity of 6,809 megawatts. Due to its large capacity, a more reasonable delimitation should assign it with a larger service scope than which delimited by the traditional VD. Let the site of GCD be g, cell of GCD be c. Unlike the traditional VD, for each adjacent site p of g and the corresponding edge e in between, the distance from p to e is no longer the same as which from e to g; rather, the distance from g to e is significantly longer than e to p. For example, as shown in Figure 15, the site of Chief Joseph power plant has weight = 0.88 while g has weight = 1, and the distance from the site of Chief Joseph to the edge between it and g is 51.71 pixels in the diagram(equivalent to 51.71 pixel *390 meters/pixel = 20,166.9 meters[1]), whereas the distance from g to the edge

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[1]Google Map has zoom levels from 0(most zoomed out) to 21 with 256x256 pixels tiles [Goob]. Here the diagram has zoom level = 8. Each zoom level has a conversion rate from pixel distance to the real world distance given the latitude of the location. With the latitude of the location of GCD = 47.96N, the estimated conversion rate is about 390 meters/pixel at zoom level = 8. [Gooc]

is 60.57 pixels(equivalent to 60.57 pixel*390 meters/pixel = 23,622.3 meters). Since all edges of the c displays the similar pattern, that is, further away from g than from the adjacent p, the area of c is larger than the corresponding cell of the traditional VD. Generalized to the whole diagram, for any two adjacent sites, the edge in between is closer to the one has larger weight. Since the area of a cell is determined by the distances from the edges to the site, the sites with larger weights(compared with the adjacent sites) have larger cells. Therefore, since power plants with larger capacities are assigned with larger service scopes, PD produces a more reasonable service scope delimitation.

Figure 14: PD in GCD area
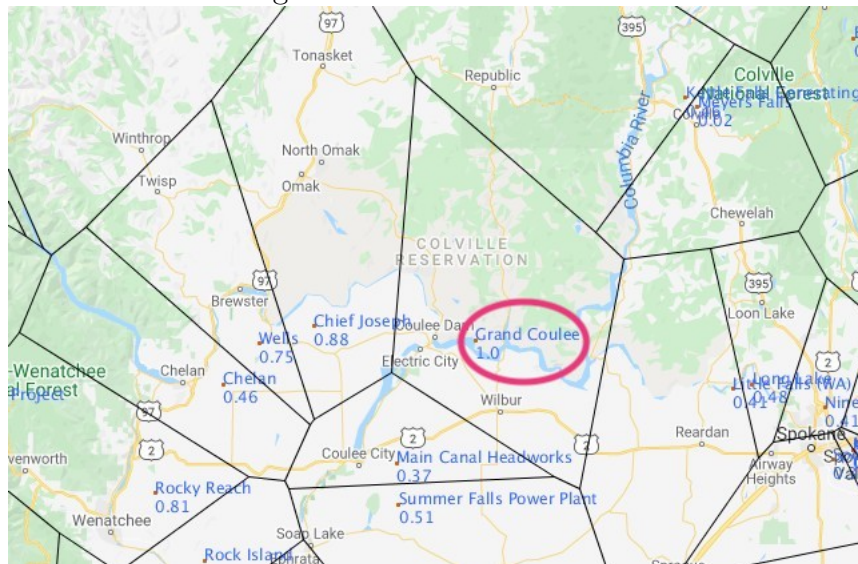
6   Summary and Future Direction

This paper applies Voronoi diagram, both the traditional and the weighted VD(power diagram), to delimit service scopes for power plants in the U.S. to optimize transmission efficiency. The traditional VD can optimize transmission efficiency since each demand point is assigned to the closest power plant; however, it does not consider the huge variation in the capacity of the power plants, which might result in short supply or oversupply in reality. The weighted VD incorporates capacity as weight, so the service scope now is co-determined
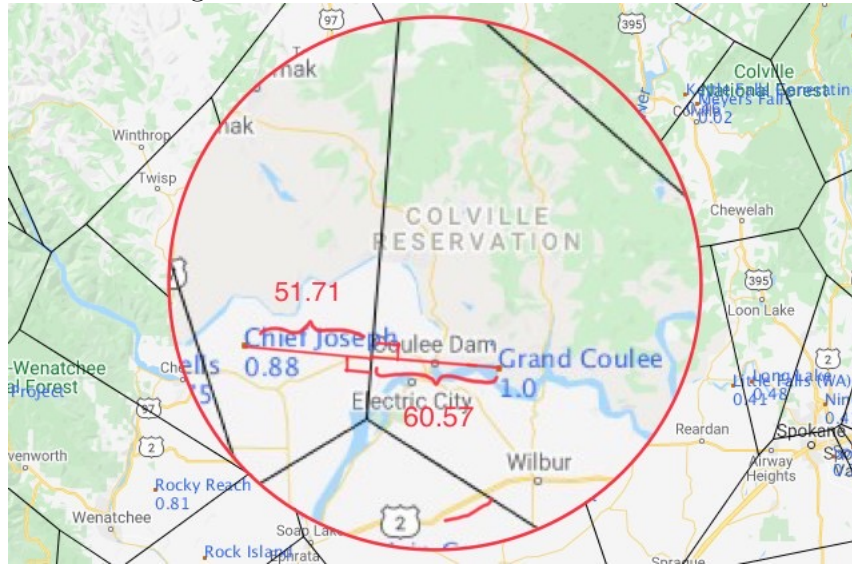
Figure 15: PD in GCD Area Zoomed-In

Figure 16: Chief Joseph power plant is 51.71 pixel distance away from the edge on its right, and GCD is 60.57 pixel distance away from the edge on its left.

by both location and capacity. In this way, power plants with larger capacities are likely to have larger service scopes, so the model produced by weighted VD is more reasonable and scientific.

However, the model is still highly idealized. Besides capacity, future work could take more practical factors into consideration, including geographical unevenness and restrictions, demographical variations such as population density and electricity consumption per capita, etc.

References

[ADMS13] Lyudmil Aleksandrov, Hristo Djidjev, Anilv Maheshwari, and Jörg-Rüdiger Sack. An approximation algorithm for computing shortest paths in weighted 3-d domains. In *Discrete Computational Geometry*, page 124–184. Springer, 2013. doi: 10.1007/s00454-013-9486-0.

[Age] United States Environmental Protection Agency. Centralized generation of electricity and its impacts on the environment. https://www.epa.gov/energy/centralized-generation-electricity-and-its-impacts-environment. Accessed: March 7, 2021.

[BDP$^+$02] Jean Daniel Boissonnat, Olivier Devillers, Sylvain Pion, Monique Teillaud, and Mariette Yvinec. Triangulations in cgal. *Computational Geometry*, 22(1):5–19, 2002. doi: 10.1016/S0925-7721(01)00054-2.

[BFH$^+$19] Logan Byers, Johannes Friedric, Roman Hennig, Aaron Kressig, Xinyue Li, Colin McCormick, and Laura Malaguzzi Valeri. A global database of power plants. Technical report, World Resources Institute, June 2019. https://www.wri.org/publication/global-power-plant-database.

[con21] Wikipedia contributors. Delaunay triangulation — Wikipedia, the free encyclopedia, 2021. https://en.wikipedia.org/w/index.php?title=Delaunay_triangulation&oldid=1012384274. Accessed: March 24, 2021.

[DMDG15] Joydeep Das, Subhashis Majumder, Debarshi Dutta, and Prosenjit Gupta. Iterative use of weighted voronoi diagrams to improve scalability in recommender systems. In *Advances in Knowledge Discovery and Data Mining*, pages 605–617, Cham, Switzerland, 2015. Springer International Publishing.

[DO11] Stayan L. Devadoss and Joseph O'Rourke. *Discrete and Computational geometry*. Princeton University Press, Princeton, NJ, first edition, 2011.

[Ele]   The Electropedia. Energy efficiency. https://www.mpoweruk.com/energy_efficiency.html. Accessed: March 7, 2021.

[ETH13]   ETH Zurich. Delaunay triangulations, 2013. https://www.ti.inf.ethz.ch/ew/Lehre/CG13/lecture/Chapter\%206.pdf. Accessed: March 24, 2021.

[Flo16]   Florian Rudolf. Topological operations, 2016. https://www.iue.tuwien.ac.at/phd/rudolf/node24.html. Accessed: 24-March-2021.

[FLZY]   Yongfeng Fan, Wenxia Liu, Jianhua Zhang, and Xu Yang. The dynamic planning of urban substation based on weighted voronoi diagram. In *2009 Asia-Pacific Power and Energy Engineering Conference*, pages 1–4.

[FM18]   Xin Feng and Alan Murray. Allocation using a heterogeneous space voronoi diagram. *Journal of Geographical Systems*, 20, July 2018. doi: 10.1007/s10109-018-0274-5.

[For86]   Steven Fortune. A sweepline algorithm for voronoi diagrams. In *Proceedings of the Second Annual Symposium on Computational Geometry*, pages 313–322, Yorktown Heights, New York, NJ, 1986. Association for Computing Machinery. doi: 10.1145/10515.10549.

[GK12]   Nicolai Guth and Philipp Klingel. Demand allocation in water distribution network modelling – a gis-based approach using voronoi diagrams with constraints. In Bhuiyan Monwar Alam, editor, *Application of Geographic Information Systems*, chapter 15. IntechOpen, Rijeka, 2012. doi: 10.5772/50014.

[Gooa]   Google. Google maps javascript api overview. https://developers.google.com/maps/documentation/javascript/overview. Accessed: March 23, 2021.

[Goob]   Google. Map and tile coordinates. https://developers.google.com/maps/documentation/javascript/coordinates. Accessed: April 12, 2021.

[Gooc] Google. Zoom level. https://docs.mapbox.com/help/glossary/zoom-level/. Accessed: April 12, 2021.

[Kan08] James M. Kang. Voronoi diagram. In Shashi Shekhar and Hui Xiong, editors, *Encyclopedia of GIS*, pages 1232–1235. Springer US, Boston, MA, 2008. doi: $10.1007/978\text{-}0\text{-}387\text{-}35973\text{-}1_1461$.

[Kol04] Cyrus Kolahdouzan, Mohammad afnd Shahabi. Voronoi-based k nearest neighbor search for spatial network databases. In *Proceedings of the Thirtieth International Conference on Very Large Databases*, volume 30 of *VLDB '04*, page 840–851, Toronto, Canada, 2004. VLDB Endowment.

[KRE06] Heidi Koivistoinen, Minna Ruuska, and Tapio Elomaa. A voronoi diagram approach to autonomous clustering. In Ljupčo Todorovski, Nada Lavrač, and Klaus P. Jantke, editors, *Discovery Science*, pages 149–160, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[LLK$^+$12] Hui Li, Kang Li, Taehyong Kim, Aidong Zhang, and Murali Ramanathan. Spatial modeling of bone microarchitecture. In Atilla Baskurt and Robert Sitnik, editors, *Three-Dimensional Image Processing (3DIP) and Applications II*, volume 8290, pages 232 – 240. International Society for Optics and Photonics, SPIE, 2012. doi: 10.1117/12.907371.

[LLTB11] Ickjai Lee, Kyungmi Lee, and Christopher Torpelund-Bruin. Raster voronoi tessellation and its application to emergency modeling. *Geo-spatial Information Science*, 14(4):235–245, December 2011. doi: 10.1007/s11806-011-0569-x.

[LSL16] Jingna Liu, Xiaoyun Sun, and Shujuan Liu. Weighted node network voronoi diagram and its application to optimization of chain stores layout. 7:679–688, 2016. doi: 10.1007/s13042-015-0491-x.

[Mar19] Jeremi Martin. Distributed vs. centralized electricity generation: Are we witnessing a change of paradigm? an introduction to distributed generation executive summary content. Technical report, HEC Paris, May 2019.

[OBS09]   Arsuyuki Okabe, Barry Boots, and Kokichi Sugihara. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* Wiley Series in Probability and Statistics. Wiley, 2009.

[oE]   UCLA Dept. of Epidemiology. Dr. snow's report, "in Report on the Cholera Outbreak in the Parish of St. James, Westminster, during the Autumn of 1854". http://www.ph.ucla.edu/epi/snow/drsnowsrepparishstjames.html. Accessed: March 7, 2021.

[Riv19]   Philippe Rivière. D3-voronoi, October 2019. https://github.com/d3/d3-voronoi. Accessed: March 23, 2021.

[Sch98]   Alfred H. Schumann. Thiessen polygon. In *Encyclopedia of Hydrology and Lakes*, pages 648–649. Springer Netherlands, Dordrecht, Netherlands, 1998. doi: 10.1007/1-4020-4497-6$_2$20.

[Shi]   Shimizu. Takasaki city fuel gas station voronoi diagram. https://gist.github.com/shimizu/1f8c1f44e29f93a34a2be3be1db5829d. Accessed: March 23, 2021.

[shm]   shmoon. Focus and directrix. https://www.shmoop.com/conics/parabola-parts.html. Accessed: March 24, 2021.

[SPR19]   Francesco Santelli, Francesco Palumbo, and Giancarlo Ragozini. Statistical archetypal analysis for cognitive categorization. In Alessandra Petrucci, Filomena Racioppi, and Rosanna Verde, editors, *New Statistical Developments in Data Science*, pages 77–89, Cham, Switzerland, 2019. Springer International Publishing.

[Tah06]   Abdel Aziz Taha. Coloring of the self-organizing map based on class labels. Master's thesis, October 2006.

[Tra21]   Khuyan Tran. How to find the nearest hospital with a voronoi diagram, June 2021. https://towardsdatascience.com/how-to-find-the-nearest-hospital-with-voronoi-diagram-63bd6d0b7b75. Accessed: March 07, 2021.

[Wik20]   Wikipedia contributors. Power diagram — Wikipedia, the free encyclopedia, 2020. https://en.wikipedia.org/w/index.php?title=Power_diagram&oldid=984639050. Accessed: April 13, 2021.

[Wik21]   Wikipedia contributors. Convex hull — Wikipedia, the free encyclopedia, 2021. https://en.wikipedia.org/w/index.php?title=Convex_hull&oldid=1006367656. Accessed: March 24, 2021.

[WLX+20]   Ajun Wan, Yixuan Liu, Xinjue Xie, Runqiu Tu, and Xiaolei Qi. Study on spatial layout optimization of linpan settlements based on point pattern analysis. *Environment, Development and Sustainability*, October 2020. doi: 10.1007/s10668-020-01042-z.

[XSM+10]   Zheng Xinqi, Wang Shuqing, Fu Meichen, Zhao Lu, and Yang Shujia. Urban cluster layout based on voronoi diagram. In Khaled Elleithy, editor, *Advanced Techniques in Computing Sciences and Software Engineering*, pages 13–18, Dordrecht, Netherlands, 2010. Springer Netherlands.

[YCC+20]   Wenhao Yu, Yujie Chen, Zhanlong Chen, Zelong Xia, and Qi Zhou. Service area delimitation of fire stations with fire risk analysis: Implementation and case study. *International journal of environmental research and public health*, 17(6):2030, 2020. doi: 10.3390/ijerph17062030.

[YJU12]   Wilfredo F. Yushimito, Miguel Jaller, and Satish Ukkusuri. A voronoi-based heuristic algorithm for locating distribution centers in disasters. *Networks and Spatial Economics*, 12(1):21–39, March 2012. doi: 10.1007/s11067-010-9140-9.

[ZJC18]   Pengyan Zhang, Wenlong Jing, and Yunzeng Chen. Weighted voronoi diagram-based simulation and comparative analysis of ecosystem service coverage: Case study of the zhongyuan urban agglomeration. *Journal of Sensors*, 2018. doi: 10.1155/2018/7147524.

Appendix